ON THE OSTROWSKI-GRÜSS TYPE INEQUALITY FOR TWICE DIFFERENTIABLE FUNCTIONS

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Abstract. In this paper, we obtained some new Ostrowski-Grüss type inequalities contains twice differentiable functions.

1. Introduction

In [1], Ostrowski proved the following inequality.

Theorem 1. *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a mapping differentiable in the interior of I and $a, b \in I^o, a < b$. If $|f'| \leq M, \forall t \in [a, b]$, then we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) M, \tag{1.1}$$

for $x \in [a, b]$.

In the past several years there has been considerable interest in the study of Ostrowski type inequalities. In [8], Özdemir et al. proved Ostrowski's type inequalities for (α, m) −convex functions and in [9], an Ostrowski type inequality was given by Sarıkaya. However, some new types inequalities are established, for example inequalities of Ostrowski-Grüss type and inequalities of Ostrowski-Chebyshev type. In [2], Milovanovic and Pecaric gave generalization of Ostrowski's inequality and some related applications. It was for the first time that Ostrowski-Grüss type inequality was given by Dragomir and Wang in [5]. In [4], Matic et al., generalized and improved this inequality. For generalizations, improvements and recent results see the papers [2], [3], [4], [5] and [6]. Recently, in [7], Ujevic proved following theorems;

Theorem 2. *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a mapping differentiable in the interior of I and $a, b \in I^o, a < b$. If there exist constants $\gamma, \Gamma \in \mathbb{R}$ such that $\gamma \leq f'(t) \leq \Gamma, \forall t \in [a, b]$ and $f' \in L_1[a, b]$, then we have*

$$\left| f(x) - \left(x - \frac{a+b}{2}\right) \frac{f(b) - f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{(b-a)}{2} (S - \gamma) \tag{1.2}$$

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and

$$\left|f(x)-\left(x-\frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{(b-a)}{2}\left(\Gamma-S\right),\tag{1.3}$$

where $S=\frac{f(b)-f(a)}{b-a}$.

Theorem 3. *Let $f:I\to\mathbb{R}$, where $I\subset\mathbb{R}$ is an interval, be a twice continuously differentiable mapping in the interior of I with $f''\in L_2[a,b]$ and $a,b\in I^o, a<b$. Then we have*

$$\left|f(x)-\left(x-\frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\frac{(b-a)^{\frac{3}{2}}}{2\pi\sqrt{3}}\left\|f''\right\|_2,\tag{1.4}$$

for $x\in[a,b]$.

The main purpose of this paper is to prove Ostrowski-Grüss type inequalities similar to above but now for involving twice differentiable mappings.

2. Main Results

Theorem 4. *Let $f:I\to\mathbb{R}$, where $I\subset\mathbb{R}$ is an interval, be a twice differentiable mapping in the interior of I and $a,b\in I^o, a<b$. If there exist constants $\gamma,\Gamma\in\mathbb{R}$ such that $\gamma\leq f''(t)\leq\Gamma, \forall t\in[a,b]$ and $f''\in L_2[a,b]$, then we have*

$$\begin{aligned}&\left|f(x)-xf'(x)-\frac{a^2f'(a)-b^2f'(b)}{2(b-a)}\right.\\&\left.-\left(\frac{x^2}{2}-\frac{a^2+ab+b^2}{3}\right)\frac{f'(b)-f'(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)dt\right|\\\leq&\ \frac{(b-a)^2}{3}\left(S-\gamma\right)\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}&\left|f(x)-xf'(x)-\frac{a^2f'(a)-b^2f'(b)}{2(b-a)}\right.\\&\left.-\left(\frac{x^2}{2}-\frac{a^2+ab+b^2}{3}\right)\frac{f'(b)-f'(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)dt\right|\\\leq&\ \frac{(b-a)^2}{3}\left(\Gamma-S\right)\end{aligned}\tag{2.2}$$

where $S=\frac{f'(b)-f'(a)}{b-a}$.

Proof. We can define $K(x,t)$ as following

$$K(x,t)=\begin{cases}\frac{t}{2}\left(t-2a\right), & t\in[a,x]\\[2mm]\frac{t}{2}\left(t-2b\right), & t\in(x,b]\end{cases}$$

Integrating by parts, we have

$$\begin{aligned}
& \frac{1}{b-a}\int_a^b K(x,t)f''(t)dt \\
(2.3) \qquad = & \frac{1}{b-a}\left[\int_a^x \frac{t}{2}(t-2a)\,f''(t)dt + \int_x^b \frac{t}{2}(t-2b)\,f''(t)dt\right] \\
= & \; xf'(x) - f(x) + \frac{a^2 f'(a) - b^2 f'(b)}{2\,(b-a)} + \frac{1}{b-a}\int_a^b f(t)dt
\end{aligned}$$

It is easy to see that

$$\frac{1}{b-a}\int_a^b K(x,t)dt = \frac{x^2}{2} - \frac{a^2+ab+b^2}{3} \tag{2.4}$$

and

$$\int_a^b f''(t)dt = f'(b) - f'(a) \tag{2.5}$$

Using (2.3), (2.4) and (2.5), we get

$$\begin{aligned}
& xf'(x) - f(x) + \frac{a^2 f'(a) - b^2 f'(b)}{2\,(b-a)} - \left(\frac{x^2}{2} - \frac{a^2+ab+b^2}{3}\right)\frac{f'(b)-f'(a)}{b-a} + \frac{1}{b-a}\int_a^b f(t)dt \\
= & \frac{1}{b-a}\int_a^b K(x,t)f''(t)dt - \frac{1}{(b-a)^2}\int_a^b f''(t)dt\int_a^b K(x,t)dt
\end{aligned}$$

We denote

$$R_n(x) = \frac{1}{b-a}\int_a^b K(x,t)f''(t)dt - \frac{1}{(b-a)^2}\int_a^b f''(t)dt\int_a^b K(x,t)dt$$

If we write $R_n(x)$ as following with $C \in \mathbb{R}$ which is an arbitrary constant, then we have

$$R_n(x) = \frac{1}{b-a}\int_a^b (f''(t) - C)\left[K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\right]dt \tag{2.6}$$

We know that

$$\int_a^b \left[K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\right]dt = 0 \tag{2.7}$$

So, if we choose $C = \gamma$ in (2.6). Then we get

$$R_n(x) = \frac{1}{b-a}\int_a^b (f''(t) - \gamma)\left[K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)ds\right]dt$$

and

$$|R_n(x)| \leq \frac{1}{b-a} \max_{t\in[a,b]} \left| K(x,t) - \left(\frac{x^2}{2} - \frac{a^2+ab+b^2}{3} \right) \right| \int_a^b |f''(t) - \gamma| \, dt \tag{2.8}$$

Since

$$\max_{t\in[a,b]} \left| K(x,t) - \left(\frac{x^2}{2} - \frac{a^2+ab+b^2}{3} \right) \right| = \frac{(b-a)^2}{3}$$

and

$$\begin{aligned} \int_a^b |f''(t) - \gamma| \, dt &= f'(b) - f'(a) - \gamma(b-a) \\ &= (S-\gamma)(b-a) \end{aligned}$$

from (2.8), we have

$$|R_n(x)| \leq \frac{(b-a)^2}{3} (S-\gamma) \tag{2.9}$$

which gives (2.1).

Second, if we choose $C = \Gamma$ in (2.6) and by a similar argument we get

$$|R_n(x)| \leq \frac{1}{b-a} \max_{t\in[a,b]} \left| K(x,t) - \left(\frac{x^2}{2} - \frac{a^2+ab+b^2}{3} \right) \right| \int_a^b |f''(t) - \Gamma| \, dt \tag{2.10}$$

and

$$\begin{aligned} \int_a^b |f''(t) - \Gamma| \, dt &= \Gamma(b-a) - f'(b) + f'(a) \\ &= (\Gamma - S)(b-a) \end{aligned} \tag{2.11}$$

From (2.10) and (2.11), we get (2.2). □

Theorem 5. *Let $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, be a twice continuously differentiable mapping in the interior of I with $f'' \in L_2[a,b]$ and $a, b \in I^o, a < b$. Then we have*

$$\begin{aligned} &\left| f(x) - xf'(x) - \frac{a^2 f'(a) - b^2 f'(b)}{2(b-a)} \right. \\ &\quad \left. - \left(\frac{x^2}{2} - \frac{a^2+ab+b^2}{3} \right) \frac{f'(b) - f'(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ &\leq \frac{(b-a)^2}{3} \left(S - f''\left(\frac{a+b}{2} \right) \right) \end{aligned} \tag{2.12}$$

where $S = \frac{f'(b)-f'(a)}{b-a}$.

Proof. $R_n(x)$ be defined as above, we can write

$$R_n(x) = \frac{1}{b-a} \int_a^b (f''(t) - C) \left[K(x,t) - \frac{1}{b-a} \int_a^b K(x,s) ds \right] dt \tag{2.13}$$

If we choose $C = f''\left(\frac{a+b}{2}\right)$, we get

$$|R_n(x)| \leq \frac{1}{b-a} \max_{t\in[a,b]} \left| K(x,t) - \left(\frac{x^2}{2} - \frac{a^2+ab+b^2}{3}\right)\right| \int_a^b \left| f''(t) - f''\left(\frac{a+b}{2}\right)\right| dt$$

By a simple computation, we get the required result. □

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